Integration of irrational functions

Integration of irrational functions

The following are some of the standard forms of irrational functions.

Type I

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

Formulae used in this topic

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + C = \log\left(x+\sqrt{x^2-a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a) + C = \log\left(x+\sqrt{a^2+x^2}\right) + C$$

Example:

(i)Evaluate $\int \frac{1}{\sqrt{[x^2-x-2]}}\,dx$

Solution:

Consider, $[x^2-x-2] = x^2 - x + \frac{1}{4} - \frac{9}{4}$

$$= \left[\left(x-\frac{1}{2}\right)^2 - \frac{9}{4}\right] \quad = \left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

$$\int \frac{1}{\sqrt{[x^2-x-2]}}\,dx = \int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}}\,dx$$

$$= \log\left(x-\frac{1}{2}\right) + \sqrt{[x^2-x-2]} + C$$

(ii)Evaluate $\int \frac{1}{\sqrt{2x^2-x+5}}\,dx$

Solution:

Consider, $2x^2 - x + 5 = 2\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{5}{2}\right]$

$$= 2\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \frac{1}{16} + \frac{5}{2}\right]$$

$$= 2\left[\left(x-\frac{1}{4}\right)^2 + \left(\frac{\sqrt{39}}{4}\right)^2\right]$$

$$\int \frac{1}{\sqrt{2x^2-x+5}}\,dx = \int \frac{1}{\sqrt{2\left[\left(x-\frac{1}{4}\right)^2 + \left(\frac{\sqrt{39}}{4}\right)^2\right]}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left[\left(x-\frac{1}{4}\right)^2 + \left(\frac{\sqrt{39}}{4}\right)^2\right]}}\,dx$$

$$= \frac{1}{\sqrt{2}} \log\left(\left(x - \frac{1}{4}\right) + \frac{1}{\sqrt{2}}\sqrt{2x^2 - x + 5}\right) + C$$

TYPE II

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx$$

Put $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B$ and then proceed.

Example:

(i)Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\, dx$

Solution:

Let $x = A\frac{d}{dx}(3 - 2x - x^2) + B$

$$= A(-2 - 2x) + B$$

$$x = -2Ax + (-2A + B)$$

Equating the coefficients of x and constant terms on both sides

$$1 = -2A \Rightarrow A = -\frac{1}{2} \; and \; 0 = -2A + B \Rightarrow B = -1$$

$$= \int \frac{x}{\sqrt{3-2x-x^2}}\, dx = \frac{1}{2}\int \frac{\frac{d}{dx}(3-2x-x^2)}{\sqrt{3-2x-x^2}}\, dx - \int \frac{1}{\sqrt{3-2x-x^2}}\, dx$$

$$= \frac{1}{2} \times 2\sqrt{3 - 2x - x^2} - \int \frac{1}{\sqrt{3-2x-x^2}}\, dx$$

Now we complete square

$$3 - 2x - x^2 = 4 - 1 - 2x - x^2 = 4 - (x + 1)^2$$

$$= \int \frac{x}{\sqrt{3-2x-x^2}}\, dx = -\sqrt{3 - 2x - x^2} - \int \frac{1}{\sqrt{3-2x-x^2}}\, dx$$

$$= -\sqrt{3 - 2x - x^2} - \int \frac{1}{\sqrt{4-(x+1)^2}}\, dx$$

$$= -\sqrt{3 - 2x - x^2} - sin^{-1}\left(\frac{x+1}{2}\right) dx$$

(ii)Evaluate $\int \frac{2x-1}{\sqrt{x^2+5x+6}}\, dx$

Solution:

Given $\int \frac{2x-1}{\sqrt{x^2+5x+6}}\, dx$

$$2x - 1 = A\frac{d}{dx}(x^2 + 5x + 6) + B$$

$$2x - 1 = A(2x + 5) + B$$

Equating the coefficients of x we get

$$2 = 2A \quad \Rightarrow A = 1$$

Put $x = 0$ we get

$$-1 = 5A + B$$

$$-1 = 5 + B \Rightarrow B = -6$$

$$\therefore \int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx = \int \frac{2x-5}{\sqrt{x^2+5x+6}}\,dx + \int \frac{-6}{\sqrt{x^2+5x+6}}\,dx$$

$$= 2\sqrt{x^2+5x+6} - 6\int \frac{1}{\sqrt{\left[\left(x+\frac{5}{2}\right)^2+6-\frac{25}{4}\right]}}\,dx$$

$$= 2\sqrt{x^2+5x+6} - 6\int \frac{1}{\sqrt{\left[\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}}\,dx$$

$$= 2\sqrt{x^2+5x+6} - 6\log\left[\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right] + C$$

TYPE III

$$\int \sqrt{ax^2+bx+c}\,dx$$

Formulae used

$$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right) + C$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right) + C$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right) + C$$

Example:

(i)Evaluate $\int \sqrt{x^2+2x+5}\,dx$

Solution:

Consider $\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{(x+1)^2+4}\,dx$

$$= \frac{x+1}{2}\sqrt{x^2+2x+5} + \frac{4}{2}\log\left((x+1)+\sqrt{x^2+2x+5}\right) + C$$

(ii)Evaluate $\int \sqrt{5+4x-x^2}\,dx$

Solution:

Consider $5+4x-x^2 = 9-(x^2-4x+4)$

$$= 9-(x-2)^2$$

Hence $\int \sqrt{5+4x-x^2}\,dx = \int \sqrt{9-(x-2)^2}\,dx$

$$= \frac{9}{2}sin^{-1}\left(\frac{x-2}{3}\right) + \left(\frac{x-2}{2}\right)\sqrt{5+4x-x^2} + C$$

Type IV

$$\int (\boldsymbol{px+q})\sqrt{\boldsymbol{ax^2+bx+c}}\ \boldsymbol{dx}$$

Put $px+q = A\,d(ax^2+bx+c) + B$

$= A\int\sqrt{ax^2+bx+c}\ d\,(ax^2+bx+c) + B\int\sqrt{ax^2+bx+c}\ dx$ and then proceed.

Example:

(i)Evaluate $\int (\boldsymbol{3x-2})\sqrt{\boldsymbol{x^2+x+1}}\ \boldsymbol{dx}$

Solution:

Given $\int(3x-2)\sqrt{x^2+x+1}\ dx$

Put $3x-2 = A\,d(x^2+x+1) + B$

$3x-2 = A\,(2x+1) + B$

Equating the coefficients of x we get

$$3 = 2A \quad \Rightarrow A = \frac{3}{2}$$

Put $x = 0$ we get

$$-2 = A+B = \frac{3}{2}+B$$

$$B = -2-\frac{3}{2} = \frac{-7}{2}$$

$\int(3x-2)\sqrt{x^2+x+1}\ dx$

$$= \frac{3}{2}\int\sqrt{x^2+x+1}\,d\,(x^2+x+1) + \left(\frac{-7}{2}\right)\int\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\frac{(x^2+x+1)^{3/2}}{3/2} - \frac{7}{2}\sqrt{\left[\left(x+\frac{1}{2}\right)^2+1-\frac{1}{4}\right]}\,dx$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2}\sqrt{\left[\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]}\ dx$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2}\left[\frac{x+\frac{1}{2}}{2}\sqrt{x^2+x+1} + \frac{\left(\frac{3}{4}\right)}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right] + C$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1} + \frac{3}{8}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right] + C$$

TYPE V

$$\int\frac{\boldsymbol{dx}}{(\boldsymbol{px+q})\sqrt{\boldsymbol{ax^2+bx+c}}}$$

Put $px+q = \frac{1}{t}$ and then proceed

Example:

(i)Evaluate $\int\frac{\boldsymbol{dx}}{(\boldsymbol{x-1})\sqrt{\boldsymbol{x^2+2x-8}}}\ \boldsymbol{dx}$

Solution:

Given $\int \frac{dx}{(x-1)\sqrt{x^2+2x-8}}\ dx$

Put $x-1 = \frac{1}{t}$; $x = 1+\frac{1}{t}$; $dx = \frac{-1}{t^2}\ dt$

$$= \int \frac{1}{\frac{1}{t}\sqrt{\left[\left(1+\frac{1}{t}\right)^2+2\left(1+\frac{1}{t}\right)-8\right]}}\left(\frac{-1}{t^2}\right)\ dt$$

$$= -\int \frac{1}{t\sqrt{\frac{(1+t)^2}{t^2}+2\frac{(t+1)}{t}-8}}\ dt$$

$$= -\int \frac{1}{\sqrt{(1+t)^2+2t\ (t+1)-8t^2}}\ dt$$

$$= -\int \frac{1}{\sqrt{1+t^2+2t+2t^2+2t-8t^2}}\ dt$$

$$= -\int \frac{1}{\sqrt{1+4t-5t^2}}\ dt$$

$$= -\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{-t^2+\frac{4}{5}t+\frac{1}{5}}}dt$$

$$= -\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{-t^2+\frac{4}{5}t+\left(\frac{2}{5}\right)^2-\left(\frac{2}{5}\right)^2+\frac{1}{5}}}dt$$

$$= -\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{\frac{9}{25}-\left(t-\frac{2}{5}\right)^2}}\ dt$$

$$= -\frac{1}{\sqrt{5}}sin^{-1}\left(\frac{t-\frac{2}{5}}{\frac{3}{5}}\right)+C$$

$$= -\frac{1}{\sqrt{5}}sin^{-1}\left(\frac{5t-2}{\frac{3}{5}}\right)+C$$

$$= -\frac{1}{\sqrt{5}}sin^{-1}\left(\frac{\frac{5}{x-1}-2}{3}\right)+C$$

Type VI

$$\int \frac{\boldsymbol{dx}}{(\boldsymbol{ax^2+b})\sqrt{\boldsymbol{cx^2+d}}}$$

Put $x = \frac{1}{t}$ and then proceed.

Example:3.82

(i)Prove that $\int_0^1 \frac{dx}{1+x^2\sqrt{x^2+2}} = \pi/6$

Solution:

Given $\int \frac{dx}{1+x^2\sqrt{x^2+2}}$

Put $x = \frac{1}{t}$, $\quad dx = \frac{-1}{t^2}\ dt$

$$\int \frac{dx}{1+x^2\sqrt{x^2+2}} = \int \frac{\left(\frac{-1}{t^2}\right)dt}{\left(1+\frac{1}{t^2}\right)\sqrt{\frac{1}{t^2}+2}}$$

$$= \int \frac{-dt}{\left(t^2+1\right)\sqrt{\frac{1+2t^2}{t^2}}}$$

Put $u^2 = 1 + 2t^2$

$$= \int \frac{-t\,dt}{(t^2+1)\sqrt{1+2t^2}}$$

$2udu = 4t\,dt$

$$= \int \frac{-\frac{u}{2}du}{\left(\frac{u^2+1}{2}\right)\sqrt{1+u^2-1}}udu$$

$\frac{u}{2}\,du = t\,dt$

$$= \int \frac{-\frac{u}{2}du}{\left(\frac{u^2+1}{2}\right)u}$$

$u^2 = 1 + 2t^2$

$$= \int \frac{-\,du}{u^2+1}$$

$t^2 = \frac{u^2-1}{2}$

$$= -\int \frac{du}{u^2+1}$$

$1 + t^2 = 1 + \frac{u^2-1}{2}$

$$= -tan^{-1}u$$

$= \frac{u^2+1}{2}$

$$= tan^{-1}\sqrt{1+2t^2} = -tan^{-1}\sqrt{1+\frac{2}{x^2}}$$

$$\int_0^1 \frac{dx}{1+x^2\sqrt{x^2+2}} = \left[-tan^{-1}\sqrt{1+\frac{2}{x^2}}\right]_0^1$$

$$= \left(-tan^{-1}\sqrt{3}\right) - (-tan^{-1}\infty)$$

$$= \left(-\frac{\pi}{3}\right) + \frac{\pi}{2} = \frac{\pi}{6}$$

(ii)Evaluate: $\int \frac{dx}{x^2\sqrt{4+x^2}}$

Solution:

Given $\int \frac{dx}{x^2\sqrt{4+x^2}}$

Put $x = \frac{1}{t} \quad \Rightarrow \quad dx = -\frac{1}{t^2}\,dt$

$$\int \frac{dx}{x^2\sqrt{4+x^2}} = \int \frac{\frac{-1}{t^2}\,dt}{\frac{1}{t^2}\sqrt{4+\frac{1}{t^2}}}$$

$$= -\int \frac{t\,dt}{\sqrt{4t^2+1}}$$

$$= -\frac{1}{8}\int \frac{8t\,dt}{\sqrt{4t^2+1}}$$

$$= -\frac{1}{8}\int \frac{d\,(4t^2+1)}{\sqrt{4t^2+1}}$$

$$= -\frac{1}{8}\left[2\sqrt{4t^2+1}\right]$$

$$= -\frac{1}{4}\sqrt{\frac{4}{x^2}+1}$$

$$= -\frac{\sqrt{4+x^2}}{4x}$$

Type VII

An expression involves only one irrational quantity of the form $(ax+b)^{1/n}$ or $x^{1/n}$, then put $+b = t^n$, where n is the L.C.M of denominators of the various fractional powers.

Example:

(i)Evaluate $\int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}}$

Solution:

Given $\int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}}$

Since L.C.M of 2 and 3 is 6.

Let $1+x = t^6 \quad \Rightarrow \quad dx = 6t^5\,dt$

$$\int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}} = \int \frac{6\,t^5}{t^3-t^2}\,\mathrm{dt} = \int \frac{6\,t^5}{t^2(t-1)}\,\mathrm{dt}$$

$$= \int \frac{6\,t^3}{(t-1)}\,\mathrm{dt}$$

$$= 6\int \frac{(t^3-1+1)}{t-1}\,dt$$

$t^3-1^3=(t-1)(t^2+t+1)$

$$= 6\int\left[\frac{t^3-1}{t-1}+\frac{1}{t-1}\right]dt$$

$$= 6\int\left\{(t^2+t+1)+\frac{1}{t-1}\right\}dt$$

$$= 6\left[\frac{t^3}{3}+\frac{t^2}{2}+t+\log(t-1)\right]$$

$$= 2(1+x)^{1/2}+3(1+x)^{1/3}+6(1+x)^{1/6}+6\log\left\{\left((1+x)^{1/6}\right)-1\right\}$$

(ii)Evaluate $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Solution:

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx = \int \frac{1}{x^{1/2}-x^{1/3}}\,dx$$

Since L.C.M of 2 and 3 is 6.

Let $u^6 = x \qquad 6u^5du = dx$

$$\int \frac{1}{x^{1/2}-x^{1/3}}\,dx = \int \frac{1}{u^3-u^2}\;6u^5d$$

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

$$= \int \frac{1}{u^2(u-1)}\;6u^5du$$

$$= \int \frac{6u^3}{u-1}\, du$$

$$= 6\int \frac{u^3-1+1}{u-1}\, du$$

$$= 6\int \left[\frac{(u-1)(u^2+u+1)}{u-1} + \frac{1}{u-1}\right] du$$

$$= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u\right] + 6\log(u-1) + C$$

$$= 2u^3 + 3u^2 + 6u + 6\log(u-1) + C \quad \text{where } u = x^{1/6}$$

$$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\log\left(\sqrt[6]{x} - 1\right) + C$$

Type VIII

$$\int \frac{1}{\sqrt{(x-a)(x-b)}} \text{ (or)} \int \sqrt{\frac{x-a}{b-x}}, b > a$$

Put $x = acos^2\theta + bsin^2\theta$

$$= \int (u^2 + u + 1)\, du + 6\int \frac{1}{u-1}\, du$$

Example:

(i)Evaluate $\int \sqrt{\frac{5-x}{x-2}}\, dx$

Solution:

Let $x = 2sin^2\theta + 5cos^2\theta$ $\quad dx = (4\sin\theta\cos\theta - 10\cos\theta\sin\theta)d\theta$

$$dx = (-6\sin\theta\cos\theta)d\theta$$

$$\therefore 5 - x = 5(sin^2\theta + cos^2\theta) - (2sin^2\theta + 5cos^2\theta) = 3sin^2\theta$$

$$and\ x - 2 = 2sin^2\theta + 5cos^2\theta - 2(sin^2\theta + cos^2\theta) = 3\,cos^2\theta$$

$$\int \sqrt{\frac{5-x}{x-2}}\, dx = \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\cos\theta}(-6\sin\theta\cos\theta\, d\theta)$$

$$= -6\int sin^2\theta\, d\theta$$

$$= -6\int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= -3\left[\theta - \frac{\sin 2\theta}{2}\right]$$

$$= -3\theta + \frac{3}{2}\sin 2\theta$$

$$= -3\,sin^{-1}\left(\sqrt{\frac{5-x}{3}}\right) + 3\sin\theta\cos\theta$$

$$= -3\,sin^{-1}\left(\sqrt{\frac{5-x}{3}}\right) + 3\sqrt{\frac{5-x}{3}}\sqrt{\frac{x-2}{3}}$$

$$= -3\,sin^{-1}\left(\sqrt{\frac{5-x}{3}}\right) + \sqrt{(5-x)(x-2)}$$

(ii)Evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ $\quad \boldsymbol{\beta} > \alpha$

Solution:

Given $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ $\quad \beta > \alpha$

Put $x = \alpha\, sin^2\theta + \beta\, cos^2\theta \quad \Rightarrow dx = (2\,\alpha \sin\theta\cos\theta - 2\beta\cos\theta\sin\theta)\, d\theta$

$\Rightarrow dx = 2\,(\alpha - \beta)\sin\theta\cos\theta\; d\theta$

$\therefore x - \alpha = (\alpha sin^2\theta + \beta cos^2\theta) - \alpha\,(sin^2\theta + cos^2\theta) = (\beta - \alpha)cos^2\theta$

$and\ \beta - x = \beta(sin^2\theta + cos^2\theta) - (\alpha sin^2\theta + \beta cos^2\theta) = (\beta - \alpha)\, sin^2\theta$

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} = \int \frac{2\,(\alpha-\beta)\sin\theta\,\cos\theta\, d\theta}{\sqrt{(\beta-\alpha)\cos^2\theta\,(\beta-\alpha)\, sin^2\theta}}$$

$$= 2\,(\alpha - \beta)\int \frac{d\theta}{\sqrt{(\beta-\alpha)^2}}$$

$$= -\frac{2(\alpha-\beta)}{\beta-\alpha}\int d\theta$$

$$= -2\int d\theta$$

$$= -2\theta$$

$$= -2\, sin^{-1}\sqrt{\frac{\beta - x}{\beta - \alpha}}$$

Type IX

$$\int \frac{dx}{a + b\,cosx}\ \ \boldsymbol{or}\ \ \int \frac{dx}{a + b\,sinx}\ \ \textbf{or}\ \int \frac{dx}{a\,sinx + b\,cosx + C}$$

Put $t = tan\, {}^{x}/_{2} \ \Rightarrow\ dt = \frac{1}{2} sec^2\, {}^{x}/_{2}\, dx$

$= \frac{1}{2}\left[1 + tan^2\, {}^{x}/_{2}\right]dx$

$= \frac{1}{2}(1 + t^2)dx$

$dx = \frac{2}{1+t^2}\, dt$

$sinx = \frac{2t}{1+t^2},\ cosx = \frac{1-t^2}{1+t^2}$

Example:

(i)Evaluate $\int_0^{\pi} \frac{dx}{5+3cosx}$

Solution:

Consider $\int \frac{dx}{5+3cosx}$

Put $t = tan\frac{x}{2}$

$dt = \frac{1}{2}(1 + t^2)dx$

$sinx = \frac{2t}{1+t^2}$

$cosx = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2}\, dt$

$$\int \frac{dx}{5+3cosx} = \left(\int \frac{\frac{2dt}{1+t^2}}{5+3\left(\frac{1-t^2}{1+t^2}\right)}\right)$$

$$= \int \frac{2dt}{8+2t^2} = \int \frac{dt}{4+t^2}$$

$$= \frac{1}{2}\left(tan^{-1}\left(\frac{t}{2}\right)\right)$$

$$= \frac{1}{2}\left(tan^{-1}\left(\frac{tan(x/2)}{2}\right)\right)$$

$$\int_0^{\pi} \frac{dx}{5+3cosx} = \left[\frac{1}{2}\left(tan^{-1}\left(\frac{tan(x/2)}{2}\right)\right)\right]_0^{\pi}$$

$$= \frac{1}{2} tan^{-1}\left(\infty - \frac{1}{2}\, 0\right)$$

$$= \frac{1}{2} tan^{-1}\infty$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Type X

Example:

Prove that $\int_0^{\pi} \frac{dx}{a^2cos^2x+b^2sin^2x} = \frac{\pi}{ab}$

Solution:

Consider $\int \frac{dx}{a^2cos^2x+b^2sin^2x}$

$$= \int \frac{dx}{cos^2x\,[a^2+b^2tan^2x]}$$

$$= \int \frac{sec^2xdx}{a^2+b^2tan^2x}$$

put $t = tanx \;\Rightarrow dt = sec^2xdx$

$$= \int \frac{dt}{a^2+b^2t^2}$$

$$= \int \frac{dt}{b^2\left[t^2+\left(\frac{a}{b}\right)^2\right]}$$

$$= \frac{1}{ab}\left(tan^{-1}\left(\frac{bt}{a}\right)\right)$$

$$= \frac{1}{ab}\left(tan^{-1}\left(\frac{b}{a}\, tanx\right)\right)$$

$$\int_0^{\pi} \frac{dx}{a^2cos^2x+b^2sin^2x} = 2\int_0^{\pi/2} \frac{dx}{a^2cos^2x+b^2sin^2x}$$

$$= 2\left[\frac{1}{ab}\left(tan^{-1}\left(\frac{btan\,x}{a}\right)\right)\right]_0^{\pi/2}$$

$$= \frac{2}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{ab}$$